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Can billiard eigenstates be approximated by superpositions of plane waves?

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Abstract

The plane wave decomposition method (PWDM) is one of the most popular strategies for numerical solution of the quantum billiard problem. The method is based on the assumption that each eigenstate in a billiard can be approximated by a superposition of plane waves at a given energy. From the classical results on the theory of differential operators this can indeed be justified for billiards in convex domains. In contrast, in the present work we demonstrate that eigenstates of non-convex billiards, in general, cannot be approximated by any solution of the Helmholtz equation regular everywhere in $\mathbf{R}^2$ (in particular, by linear combinations of a finite number of plane waves having the same energy). From this we infer that PWDM cannot be applied to billiards in non-convex domains. Furthermore, it follows from our results that unlike the properties of integrable billiards, where each eigenstate can be extended into the billiard exterior as a regular solution of the Helmholtz equation, the eigenstates of non-convex billiards, in general, do not admit such an extension.

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1. Introduction

The quantum billiard problem in a domain $\Omega \subset \mathbf{R}^2$ is defined (in units $m = 1$) by the Helmholtz equation

$$(-\Delta - \mathrm{k}^2)\varphi(x) = 0 \qquad E = \hbar^2\mathrm{k}^2/2 \tag{1}$$

with Dirichlet boundary conditions

$$\varphi(x)|_{\partial\Omega} = 0. \tag{2}$$

The solutions E_n, φ_n of these equations determine the energy spectrum and the set of eigenstates of Ω. Studying the properties of (E_n, φ_n) in quantum billiards has became a prototype problem in 'quantum chaos'. A simple form of equations (1) and (2) suggests a natural way to solve them. First, for a given energy E one looks for a set of solutions $\{\psi^{(n)}(\mathtt{k}), n \in \mathbb{N}\}$ of the Helmholtz equation (1) in the entire plane (without any boundary conditions). For example, $\{\psi^{(n)}(\mathtt{k})\}$ can be chosen as a set of plane waves: $\{\exp(\mathrm{i}k_n x), |k_n| = \mathtt{k}, k_n \in \mathbb{R}^2\}$, or as a set of radial waves: $\{J_n(\mathtt{k}r)\exp(\mathrm{i}n\theta), n \in \mathbb{N}\}$. Then regarding $\{\psi^{(n)}(\mathtt{k})\}$ as a basis one can search for solutions of equations (1) and (2) using the ansatz

$$\varphi(x) = \sum a_i \psi^{(i)}(\mathtt{k}, x). \tag{3}$$

As a result, solving equations (1) and (2) is reduced to the algebraic problem of finding the coefficients a_i such that the linear combination (3) vanishes whenever $x \in \partial\Omega$.

The above approach has been widely used both in analytical and numerical studies of quantum billiards. In particular, it has been suggested by Berry in [1] to use expansion (3) with a Gaussian amplitude distribution to represent eigenfunctions of quantum systems with fully chaotic dynamics. This idea has been applied in numerous works to calculate various quantities associated with eigenfunctions, e.g., autocorrelation functions [1], amplitude distributions [2], statistics of nodal domains [3] etc. The same strategy can also be used for a numerical solution of equations (1) and (2). In this context it was first introduced by Heller [4] with application to the Bunimovich stadium. Since then several modifications of the method have been considered in [5–7]. Depending on the choice of the basis in the decomposition (3) one gets, in general, different numerical methods for solving equations (1) and (2). Here we will single out the basis of plane waves (PW), most often used in applications. In brief we will refer to the corresponding numerical method as the plane wave decomposition method (PWDM).

As a matter of fact, the whole strategy described above is based on the assumption that the set $\{\psi^{(n)}(\mathtt{k})\}$ furnishes an appropriate basis for the expansion of solutions of equations (1) and (2). In other words, one can use PWDM only if billiard eigenstates can be approximated by linear combinations of plane waves. That means

$$\|\varphi_n - \psi^{[N]}\|_{L^2(\Omega)} \to 0 \qquad \text{as } N \to \infty \tag{4}$$

for some sequence of the states $\psi^{[N]}$ which are of the form

$$\psi^{[N]} = \sum_{i=1}^{N} a_i\, \mathrm{e}^{\mathrm{i}k_i x} \qquad k_i \in \mathbb{R}^2 \qquad |k_i| = \mathtt{k}. \tag{5}$$

We will say that the plane wave approximation holds for a state φ_n if the limit (4) exists.

Up to now it has often been assumed that the PWDM can be applied to billiards of arbitrary shape. From the results of Malgrange [8] (see also [9]) on the theory of differential operators it is known that any solution of equation (1) regular in a convex open domain can be approximated by superpositions of plane waves with $k_i \in \mathbb{C}^2$, $|k_i| = \mathtt{k}$. Moreover, since each evanescent plane wave ($\mathrm{Im}\, k_i \neq 0$) can be approximated in a bounded domain by plane waves with real wavenumbers [10], one immediately gets:

Proposition 1. *Let $\Omega \subset \mathbf{R}^2$ be a convex bounded domain, then any solution of equation (1) regular in Ω can be approximated by plane waves.*

This shows that the eigenstates of a quantum billiard Ω admit PW approximation inside any convex domain $\Omega_1 \subset \Omega$, see figure 1(*a*). Hence, PW approximation always holds for billiard eigenstates in a local sense. Furthermore, if Ω is a convex domain one can choose Ω_1 in such a way that $\partial\Omega_1$ is arbitrarily close to $\partial\Omega$. Consequently, as a simple corollary of proposition 1 one gets:

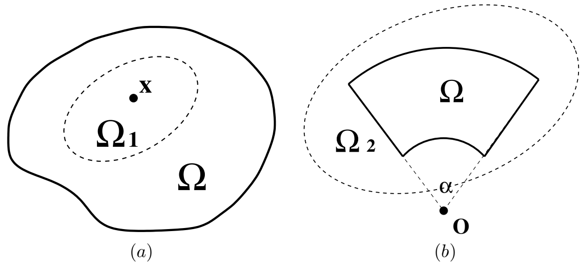

Figure 1. A typical convex domain (dashed line) where the PW approximation holds: (*a*) for a generic billiard; (*b*) for the 'cake' billiard.

Corollary 1. *Eigenstates of a convex billiard Ω can be approximated by superpositions of plane waves.*

The question naturally arises whether the same property holds for eigenstates of non-convex billiards, and thus, whether the PWDM can be actually applied to the class of non-convex billiards.

Note that there exists an important link between the PWDM and the problem of eigenstate extension in quantum billiards. We will say that an eigenstate φ_n of a billiard Ω can be extended to a domain $\Omega_2 \supset \Omega$ if there exists a regular solution $\bar{\varphi}_n(x)$; $x \in \Omega_2$ of equation (1) which coincides with $\varphi_n(x)$ inside the domain Ω. Note that such an extension (if it exists) is unique. Indeed if $\bar{\varphi}_n, \bar{\varphi}'_n$ are extensions of the same eigenfunction φ_n the difference $\bar{\varphi}_n - \bar{\varphi}'_n$ should vanish in an open domain and therefore (see, e.g., [9]) in all Ω_2. Let φ_n be an eigenstate of Ω which can be extended to a convex domain $\Omega_2 \supset \Omega$. Then it follows immediately from proposition 1 that the PW approximation holds for φ_n. The example of a billiard where each eigenstate can be continued in a convex domain is shown in figure 1(*b*). This is the 'cake' billiard whose boundary consists of two concentric circle arcs connected by two segments of radii at an angle $\alpha < \pi$. In the polar coordinates $x = (r, \theta)$ the eigenstates of the 'cake' billiard can be written explicitly as a sum of Bessel and Neumann functions:

$$\varphi_n^{(m)}(x) = \left(a_n^m J_{\nu_m}\left(\mathrm{k}_n^{(m)} r\right) + b_n^m Y_{\nu_m}\left(\mathrm{k}_n^{(m)} r\right)\right) \sin(\nu_m(\theta - \theta_0)) \qquad \nu_m = \frac{\pi m}{\alpha}.$$

Since the singularity point of $\varphi_n^{(m)}(x)$ is always at the centre O of the circle arcs it is possible to extend each eigenstate into a convex domain Ω_2, see figure 1(*b*). Accordingly, any eigenstate of the 'cake' billiard can be approximated by superpositions of PW.

On the other hand, assume that for a billiard Ω an eigenstate φ_n can be expanded in a basis $\{\psi^{(n)}\}$ (see equation (3)), where $\psi^{(i)}$ are solutions of the Helmholtz equation regular in $\mathbf{R}^2$ (e.g., plane waves). If furthermore, the corresponding sum (3) converges everywhere in $\mathbf{R}^2$ it makes sense to consider $\varphi_n(x)$ both inside and outside Ω. Such extension of $\varphi_n(x)$ into $\mathbf{R}^2$ provides simultaneously solutions for the interior Dirichlet problem (when $x \in \Omega$) and for the exterior Dirichlet problem (when $x \in \Omega^c \equiv \mathbf{R}^2/\Omega$). Based on this observation a connection (*spectral duality*) between the interior Dirichlet and the exterior scattering problems has been suggested by Doron and Smilansky in [11]. The rigorous result has been established by Eckmann and Pillet [12]. In most general form (*weak spectral duality*) it could be stated as follows: E_n is an eigenvalue of the interior problem if and only if there exists an eigenvalue $\mathrm{e}^{-\mathrm{i}\vartheta_n}$ of the exterior scattering matrix $S(E)$ such that $\vartheta_n(E) \to 2\pi$ whenever $E \to E_n$. Moreover, if $\vartheta_n(E_n) = 2\pi$ (*strong spectral duality*) then the corresponding interior eigenstate φ_n could be extended into $\mathbf{R}^2$ as an L^2 function. Therefore, if the strong form of spectral duality holds for some eigenenergy E_n then the PW approximation holds for the corresponding eigenstate φ_n.

It has been explicitly shown that the strong form of spectral duality holds for convex integrable billiards [13]. However, as has been pointed out in [12], strong spectral duality cannot hold for billiards in general.

Remark. It should be pointed out that the approximability by PW is a much weaker property than strong spectral duality. As has been explained above, strong spectral duality implies the PW approximation for the corresponding eigenstate. The opposite, however, is not true: the PW approximation for an eigenstate does not imply, in general, strong spectral duality. In fact, in [10, 12] the examples of convex billiards (in this case the approximation by PW is possible) have been constructed where the eigenstates extension into the exterior domain as L^2 functions is not possible.

2. Main results

Two different billiard maps can be associated with Ω. First, the standard billiard map Ψ corresponding to the motion of a point-like particle in the interior domain. Second, the exterior map Ψ^c which corresponds to the scattering off Ω as an obstacle, see, e.g., [14]. In order to define this map, consider the motion of a point-like particle which is injected into the domain Ω^c along a straight line l_i and undergoes specular reflection off the boundary $\partial\Omega$ from outside. Let p_i, p_{i+1} be the particle momentum before and after reflection and let $q_i \in \partial\Omega$ be the corresponding bouncing point. In general, there are two types of motions that can happen after the particle bounces off the boundary: the particle either moves along the straight line l_{i+1} and collides with $\partial\Omega$ at the next point q_{i+1} (this may happen only if Ω is a non-convex domain) or escapes to infinity along the line l_{i+1}. In the latter case we will assume that the outgoing particle is re-injected as incoming along l_{i+1} from the 'opposite' side (with the same momentum p_{i+1}) and hits the obstacle at q_{i+1}, see figure 2(a). Then the process is iterated and Ψ^c is defined as the map from (q_j, p_j) to (q_{j+1}, p_{j+1}).

It should be noted that there is an essential difference between convex and non-convex billiards. Whenever Ω is a convex domain the interior map Ψ determines the same dynamics as the exterior map Ψ^c. For any interior trajectory inside Ω there is a *dual* trajectory in Ω^c which travels through the same set of points on the boundary $\partial\Omega$, see figure 2(a). We will refer to this property as *interior–exterior duality*. In particular, for convex billiards there is one to one correspondence between the interior and exterior periodic trajectories. For each periodic trajectory γ, its continuation γ^c into the exterior domain will be the dual periodic trajectory of the exterior map. On the other hand, it is straightforward to see that in non-convex billiards interior–exterior duality breaks down. Generally, in a non-convex billiard Ω there exist interior periodic trajectories whose extension into the exterior domain intersects Ω again, see figure 2(b). Let γ be such a trajectory and let γ^c be its extension in the exterior. Note that $\gamma \cup \gamma^c$ is a union of straight lines in $\mathbf{R}^2$. Take $l \subset \gamma \cup \gamma^c$ to be a line which intersects the boundary $\partial\Omega$ at $2n, n > 1$ points (for the sake of simplicity we will always assume that $n = 2$). Then the intersection $\Omega \cap l$ is the union of two disconnected segments: $\gamma_1 \subset \gamma$ and $\bar{\gamma}_1 \subset \gamma^c$. If $\bar{\gamma}_1$ does not belong to any periodic trajectory in Ω, we will refer to γ as a *single periodic trajectory* (SPT). By definition any SPT has no dual periodic trajectory in the exterior domain. In what follows, we call a non-convex billiard Ω *generic* if it contains at least one stable (elliptic) or unstable (hyperbolic) SPT. According to this terminology the 'cake' billiard in figure 1(b) is non-generic, since all its periodic trajectories are of neutral (parabolic) type.

We call a smooth function $\psi(x)$ a *regular solution* of the Helmholtz equation if it solves equation (1) everywhere in $\mathbf{R}^2$. For a given energy E we will denote by $\mathcal{M}(E)$ the set of all

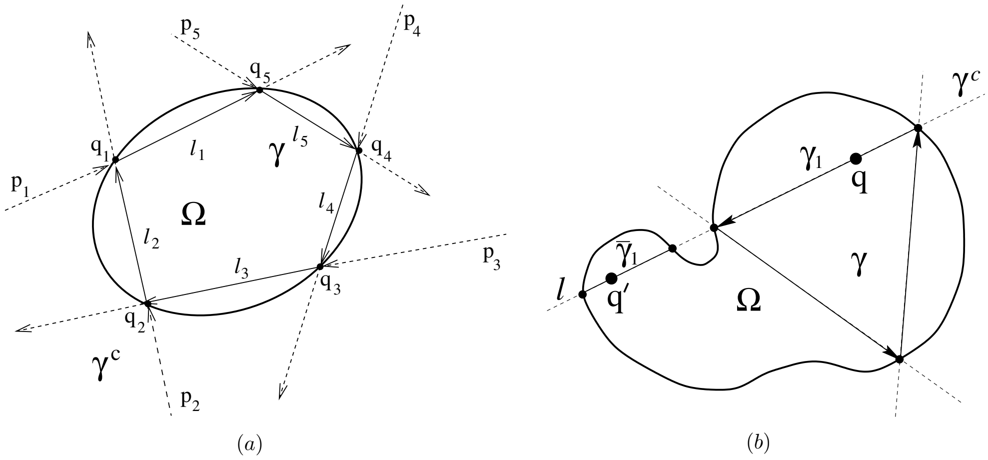

Figure 2. Breaking of interior–exterior duality. An example of a periodic trajectory (γ): (*a*) in a convex billiard; (*b*) in a non-convex billiard (SPT). Note that for a billiard in a convex domain, the continuation γ^c of γ into the exterior is a periodic trajectory of the exterior map while for non-convex billiards this interior–exterior connection breaks down.

regular solutions of equation (1) and by $\mathcal{M}_{\mathtt{PW}}(E) \subset \mathcal{M}(E)$ the subset of functions which can be represented as linear combinations of a finite number of plane waves with real wavenumbers k_i, $|k_i|^2 = 2E/\hbar^2$. In particular, $\mathcal{M}(E)$ includes convergent superpositions of plane waves (also with complex wavenumbers, i.e. evanescent modes) and radial waves with the energy E. In its crudest form the main result of the present paper can be formulated in the following way. Based on the breaking of interior–exterior duality we demonstrate that eigenstates of a generic non-convex billiard (in general) cannot be approximated by regular solutions of equation (1). Therefore, in general, PWDM fails to reproduce exact eigenstates of non-convex quantum billiards. It is worth mentioning that, in fact, this agrees with the numerical calculations performed in other works. In particular, in [6] the dependence of the accuracy of PWDM on the discretization number N of PW in equation (5) was investigated for a variety of quantum billiards. On the basis of numerical analysis it was observed there that in contrast to the case of integrable billiards, the accuracy of PWDM for the Sinai billiard and the cardioid billiard (which are of non-convex type) saturates at some point and does not improve when N is increased further.

To illustrate the main idea of our approach it is instructive to consider a non-convex billiard Ω with an elliptic SPT γ. It is well known that a sequence of quasi-modes $(\tilde{\varphi}_i, \tilde{\mathtt{k}}_i)$ associated with γ can be constructed (see, e.g., [15, 16]). Each pair $(\tilde{\varphi}_n, \tilde{\mathtt{k}}_n)$ represents an approximate solution of equations (1) and (2) such that $\tilde{\varphi}_n$ is localized along γ. Furthermore, in the absence of systematic degeneracies in the spectrum of Ω the quasi-modes $(\tilde{\varphi}_n, \tilde{\mathtt{k}}_n)$ approximate (in the L^2 sense) a sequence of real solutions $(\varphi_n, \mathtt{k}_n)$ of equations (1) and (2). For each such eigenstate φ_n let us define the corresponding Husimi function

$$H_{\varphi_n}(\mathsf{z}) = |\langle \mathsf{z}|\varphi_n\rangle|^2 \tag{6}$$

where $\langle \mathsf{z}|$ denotes a coherent state localized at the point $\mathsf{z} \in V$ of the standard billiard phase space V and $\langle\cdot|\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$. Note that in the coordinate-momentum representation V is just the set of points $\mathsf{z} = (q, p)$ such that $q \in \Omega$ and p is restricted to the energy shell $|p|^2/2 = E$. Let us consider the Husimi function (6) at the energy shell $E = E_n$, which corresponds to the eigenenergy $E_n = \hbar^2\mathtt{k}_n^2/2$ of φ_n. Then $H_{\varphi_n}(\mathsf{z})$ as a function of q

and direction of p should be localized near the line of points: $\mathsf{z} = (q, p) \in V$, where $q \in \gamma$ and p is directed along γ. Now assume that φ_n could be approximated by regular solutions of equation (1). That means for any $\epsilon > 0$ there is $\psi_\epsilon \in \mathcal{M}(E_n)$ such that $\|\varphi_n - \psi_\epsilon\| < \epsilon$, where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. In such a case one can rewrite (6) as the limit

$$H_{\varphi_n}(\mathsf{z}) = \lim_{\epsilon\to 0} |\langle \mathsf{z}|\psi_\epsilon\rangle|^2$$

of Husimi functions for regular solutions of equation (1). Furthermore, since each ψ_ϵ is an eigenstate of the free quantum evolution operator $\mathrm{e}^{-\mathrm{i}t\Delta/\hbar}$ in $\mathbf{R}^2$, one has for an arbitrary t

$$H_{\varphi_n}(\mathsf{z}) = \lim_{\epsilon\to 0} |\langle \mathsf{z}|\,\mathrm{e}^{-\mathrm{i}t\Delta/\hbar}\psi_\epsilon\rangle|^2. \tag{7}$$

(It is important to note that in contrast to equation (6), the scalar product in equation (7) should be understood as the scalar product in $L^2(\mathbf{R}^2)$.) In what follows we set q as a point at γ_1 and set p to be directed along γ_1. It is well known (see, e.g., [18, 20]) that in the semiclassical limit the free quantum evolution of coherent states is governed by the corresponding classical evolution:

$$\mathrm{e}^{-\mathrm{i}t\Delta/\hbar}|\mathsf{z}\rangle = \mathrm{e}^{\mathrm{i}tE/\hbar}|\mathsf{z}(t)\rangle + O(\hbar^\infty) \qquad \mathsf{z}(t) = (q(t), p). \tag{8}$$

(Here the symbol $O(\hbar^\infty)$ has a standard meaning: $f = O(\hbar^\infty)$ if $f = O(\hbar^\alpha)$, for any $\alpha > 0$.) Plugging (8) into equation (7) and propagating $\langle\mathsf{z}|$ up to a time t when the point $q(t) = q'$ belongs to $\bar\gamma_1$ we get

$$H_{\varphi_n}(\mathsf{z}) - H_{\varphi_n}(\mathsf{z}') = O(\hbar^\infty) \qquad \mathsf{z}' = (q', p) \in V. \tag{9}$$

This, however, contradicts the fact that $H_{\varphi_n}(\mathsf{z})$ as a function of the coordinate q should be exponentially decaying outside γ.

The above argument can be extended to the case of hyperbolic SPT γ as follows. Contrary to the elliptic case it is not possible to construct quasi-modes concentrated on hyperbolic periodic orbits. Instead, one can use a statistical approach in that case. From the results of Paul and Uribe [16] it is known that in the semiclassical limit $\hbar \to 0$, the average of the Husimi functions (6)

$$\left\langle H_{\varphi_i}(\mathsf{z})\right\rangle = \frac{1}{\#\mathcal{P}_{c\hbar}} \sum_{E_n\in\mathcal{P}_{c\hbar}} |\langle\mathsf{z}|\varphi_n\rangle|^2 \tag{10}$$

over the energy interval $\mathcal{P}_{c\hbar} = [E - c\hbar, E + c\hbar]$, $c > 0$ with the number of states $\#\mathcal{P}_{c\hbar}$ depends on whether z belongs to a periodic trajectory or not. On the other hand, as has been explained above, if each φ_n could be approximated by a regular solution of equation (1) then each $H_{\varphi_n}(\mathsf{z})$ (and therefore the average $\left\langle H_{\varphi_i}(\mathsf{z})\right\rangle$) would be (semiclassically) invariant along $\gamma_1 \cup \bar\gamma_1$ as a function of q.

The preceding discussion provides an intuitive explanation why it is impossible to approximate eigenstates of a generic non-convex billiard by a superposition of plane waves. Speaking informally our argument says that contrary to the real eigenstates of non-convex billiard Ω, any regular solution of equation (1) always 'preserves' interior–exterior duality. In what follows we consider the $L^2(\Omega)$ norm

$$\eta_n(\psi) = \|\varphi_n - \psi\| \tag{11}$$

for a solution (φ_n, E_n) of equations (1) and (2) in Ω and an arbitrary $\psi \in \mathcal{M}(E_n)$. By the definition $\eta_n(\psi)$ measures approximability of φ_n by regular solutions of the Helmholtz equation. Recall that a state φ_n is approximable by PW if

$$\inf_{\psi\in\mathcal{M}_{\mathtt{PW}}(E_n)} \eta_n(\psi) = 0.$$

Remark. Note that from proposition 1 for any $\psi \in \mathcal{M}(E_n)$ and any $\epsilon > 0$ one can always find $\psi_\epsilon \in \mathcal{M}_{\mathtt{PW}}(E_n)$ such that $|\eta_n(\psi) - \eta_n(\psi_\epsilon)| < \epsilon$. In particular, this implies

$$\eta_n^{\min} \equiv \inf_{\psi\in\mathcal{M}(E_n)} \eta_n(\psi) = \inf_{\psi\in\mathcal{M}_{\mathtt{PW}}(E_n)} \eta_n(\psi). \tag{12}$$

In other words, an eigenstate φ_n can be approximated by $\psi \in \mathcal{M}(E_n)$ if and only if it can be approximated by PW. Therefore, in what follows one can always assume without loss of generality that ψ belongs to $\mathcal{M}_{\mathtt{PW}}(E_n)$ rather than to the set $\mathcal{M}(E_n)$.

From corollary 1, $\eta_n^{\min} = 0$ for any eigenstate of a convex billiard. In contrast, in the body of the paper we show that for a generic non-convex billiard, the average of $\eta_n^{\min}$ over an energy interval is bounded from below (in the semiclassical limit) by a strictly positive constant:

Proposition 2. *Let Ω be a non-convex billiard with at least one stable or unstable SPT and let (φ_n, E_n), $n \in \mathbb{N}$ denote the eigenstates and eigenenergies of the corresponding quantum billiard. For any set of approximating functions $\{\psi_n \in \mathcal{M}(E_n), n \in \mathbb{N}\}$ the average of $\eta_n = \|\varphi_n - \psi_n\|$, $n \in \mathbb{N}$ over the energy interval $\mathcal{P}_{c\hbar} = [E - c\hbar, E + c\hbar]$, $c > 0$ satisfies*

$$\langle \eta_i \rangle > \mathcal{B}\hbar + \mathcal{O} \tag{13}$$

where $\mathcal{O} = O(\hbar^{3/2})$ and $\mathcal{B}$ is a strictly positive constant depending on the shape of Ω. Moreover, if Ω contains a SPT γ of elliptic type then (provided the spectrum of Ω has no systematic degeneracies) there exists an infinite subsequence $\mathcal{S}_\gamma = \left\{\left(\varphi_{j_m}, E_{j_m}\right), m \in \mathbb{N}\right\}$ (of a positive density, i.e. $\lim_{N\to\infty} \frac{\#\{j_m | j_m < N\}}{N} > 0$) such that for any $(\varphi_n, E_n) \in \mathcal{S}_\gamma$ and any regular solution $\psi \in \mathcal{M}(E_n)$

$$\eta_n(\psi) > \mathcal{C}_\gamma + \mathcal{O}' \tag{14}$$

where $\mathcal{O}' = O(\hbar^{1/2})$ and $\mathcal{C}_\gamma$ is a strictly positive constant which depends only on the geometrical properties of γ.

From (13) and (14) one immediately obtains the corollary:

Corollary 2. *For a generic non-convex billiard Ω there exists an infinite subsequence of eigenstates $\left\{\varphi_{j_n}, n \in \mathbb{N}\right\}$ such that: (1) $\eta_{j_n}^{\min} > 0$; (2) φ_{j_n} cannot be extended into the domain Ω^c (as a regular solution of equation (1)).*

Obviously, this implies the following properties of a generic non-convex billiard:

- in general, eigenstates of non-convex billiards do not admit approximation by PW and PWDM cannot be used in that case;
- the spectral duality for a generic non-convex billiard holds only in the weak form.

The paper is organized as follows. In the next section we collect several necessary facts about coherent states. In section 4 the case of elliptic SPT is considered. First, using the coherent states we construct a family of quasi-modes $(\tilde{\varphi}_n, \tilde{E}_n)$ associated with such trajectories. Then, we show that the lower bound (14) holds for the eigenstates φ_n approximated by $\tilde{\varphi}_n$. The case of hyperbolic SPT is considered in section 5. Here we use the results of Paul and Uribe to estimate the average $\langle \eta_i \rangle$ over an energy interval. Finally, in section 6 we discuss our results and consider possible generalizations.

3. Coherent states

3.1. Definition of coherent states

Coherent states were introduced already in the beginning of quantum mechanics and have been used in many areas since then. The basic idea is to built a complete set of vectors of Hilbert space localized in the phase space in both q and p directions at the scale $\sqrt{\hbar}$. The standard example of such states in $\mathbf{R}^d$ is given by the Gaussians:

$$u_{\mathsf{z}}^{\sigma}(x) = (\det \operatorname{Im} \sigma)^{\frac{1}{4}} (\hbar\pi)^{-\frac{d}{4}} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}[\langle p, x-q\rangle + \frac{1}{2}\langle x-q\sigma(x-q)\rangle]} \qquad \mathsf{z} = (q, p) \tag{15}$$

where angle brackets denote the scalar product in $\mathbf{R}^d$ and σ is an arbitrary $d \times d$ complex matrix with strictly positive imaginary part $\operatorname{Im}\sigma = \frac{1}{2\mathrm{i}}(\sigma - \sigma^*)$. Note that u_{z}^{σ} is a minimum-uncertainty state centred in the phase space at the point $\mathsf{z} = (q, p)$ and localized around z in an elliptic region determined by σ. In the present work we will consider a slightly more general class of coherent states. (For a more general definition of coherent states see, e.g., [16].) Let $\rho_q^{\varepsilon}(\cdot)$ be a C_0^{∞} function in $\mathbf{R}^d$ equal to one in a neighbourhood of the point q and zero outside the sphere of radius ε centred at q. A coherent state at $\mathsf{z} = (q, p)$ is the vector

$$\phi_{\mathsf{z}}^{\sigma}(x) = \rho_q^{\varepsilon}(x) u_{\mathsf{z}}^{\sigma}(x). \tag{16}$$

It is easy to see that the coherent states (16) are semiclassicaly orthogonal:

$$\left\|\phi_{\mathsf{z}}^{\sigma}\right\|^2 = 1 + O(\hbar) \qquad \left\langle \phi_{\mathsf{z}}^{\sigma} \middle| \phi_{\mathsf{z}'}^{\sigma} \right\rangle = O(\hbar^{\infty}) \qquad \text{if } \mathsf{z} \neq \mathsf{z}'. \tag{17}$$

The role of the cut-off $\rho_q^{\varepsilon}(x)$ is rather technical, it allows us to define coherent states inside compact domains. To use the vectors (16) as coherent states inside a billiard domain Ω one needs that

$$\operatorname{supp}\left[\rho_q^{\varepsilon}(x)\right] \subset \Omega. \tag{18}$$

3.2. Propagation of coherent states

An important property of coherent states is that their quantum evolution in the semiclassical limit is completely determined by the corresponding classical evolution. Let $\mathsf{H} = -\hbar^2\Delta/2 + v(x)$ be the operator of symbol $\mathcal{H} = p^2/2 + v(x)$ inducing the flow $\Psi^t : V \to V$ on the phase space V. Then, as is well known (see, e.g., [18]), for any time t the propagation of the coherent state $\phi_{\mathsf{z}}^{\sigma}$ localized at $\mathsf{z} \in V$ is given by

$$\mathrm{e}^{-\mathrm{i}t\mathsf{H}/\hbar}\phi_{\mathsf{z}}^{\sigma} = \mathrm{e}^{\mathrm{i}(S(t)/\hbar + \mu(t))}\phi_{\mathsf{z}(t)}^{\sigma(t)} + O(\hbar^{1/2}) \tag{19}$$

where $S(t) = \int_0^t (p\dot{q} - \mathcal{H}(p, q))\,\mathrm{d}t$ is the classical action along the path $\mathsf{z}(t)$ and $\mu(t)$ is the Maslov index. The parameters $\mathsf{z}(t) = \Psi^t \cdot \mathsf{z}$, $\sigma(t) = D\Psi^t \cdot \sigma$ in equation (19) are determined by the evolution of the initial data z, σ under the flow $\Psi^t : \mathsf{z} \to \mathsf{z}(t)$ and its derivative

$$D\Psi^t : \sigma \to \sigma(t) = \frac{a\sigma + b}{c\sigma + d} \tag{20}$$

where $d \times d$ matrices a, b, c, d are the components of $D\Psi^t$ in a given coordinate system:

$$D\Psi^t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is convenient to chose two of the $2d$ coordinates in the phase space V to be along the flow and along the line orthogonal to the energy surface. Then the matrix σ can be decomposed into $\sigma = \sigma^0 \oplus \sigma^1$, where the scalar part σ^0 corresponds to the above two directions and $(d-1) \times (d-1)$ matrix σ^1 corresponds to the orthogonal subspace. It is straightforward to

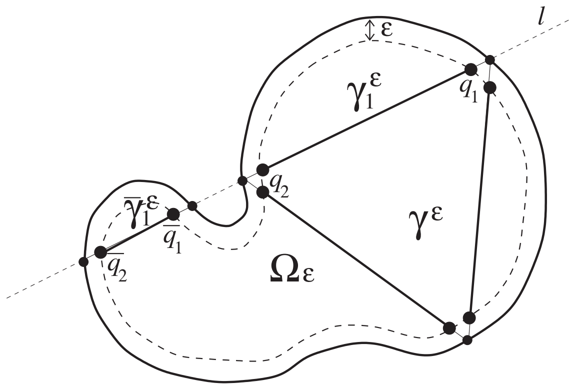

Figure 3. Definitions of inner domain $\Omega_\varepsilon \subset \Omega$ (the dashed line indicates the boundary of Ω_ε) and restrictions of $\gamma, \gamma_1, \bar{\gamma}_1$ to Ω_ε.

see that in such a basis $D\Psi^t$ acts separately on σ^1 and σ^0. In particular, $D\Psi^t \cdot \sigma^0$ is given by a linear transformation:

$$D\Psi^t \cdot \sigma^0 = \frac{\sigma^0}{u\sigma^0 + 1}. \tag{21}$$

In the present paper we will use the above results for two types of two-dimensional flows: free evolution on $\mathbf{R}^2$ under the Hamiltonian H_0 $(v(x) = 0)$ and the evolution induced by the billiard Hamiltonian H_Ω $(v(x) = 0$ if $x \in \Omega$ and $v(x) = \infty$ otherwise). Let us consider in some detail the evolution of coherent states in billiards. Set Ω as the billiard domain. We will denote the billiard flow by $\Psi_\Omega^t : V \to V$, whose action is on the standard phase space V of Ω. It should be pointed out that one can use the coherent states (16) for the point $\mathsf{z} = (q, p) \in V$ only if q is sufficiently far away from the boundary $\partial\Omega$. Indeed, to satisfy condition (18) q has to be at a distance larger than ε from the boundary. For the sake of simplicity, we will not consider a generalized class of coherent states defined in the whole domain Ω, rather we will use the states (16) but only for the interior points of Ω. For this purpose let us define the inner domain $\Omega_\varepsilon \subset \Omega$ which contains all the points q of Ω such that the distance between q and $\partial\Omega$ is larger then ε: $\mathrm{dist}(q, \partial\Omega) \geqslant \varepsilon$, see figure 3. In what follows, we will fix ε to be small compared to the linear sizes of the billiard (but large compared to $\hbar^{1/2}$) and consider the coherent states propagating under the condition that at the initial moment $t_1 = 0$ and the final moment $t_2 = t$, the points $\mathsf{z}(0), \mathsf{z}(t)$ belong to the domain Ω_ε. Whenever this condition is fulfilled one can use formula (19), where the states $\phi_{\mathsf{z}}^\sigma, \phi_{\mathsf{z}(t)}^{\sigma(t)}$ are both of the form (16). (For the Dirichlet boundary conditions one should also multiply the right-hand side of equation (19) by the factor $\mathrm{e}^{\mathrm{i}\pi n(t)}$, where $n(t)$ is the number of reflections at the billiard boundary along the classical trajectory. To simplify notation, we will always include the 'boundary' phase $\pi n(t)$ in the coefficient $\mu(t)$.) Furthermore, if $q(t) \in \Omega_\varepsilon$ for all $t \in [t_1, t_2]$ (i.e. there is no collision with the boundary between the times t_1 and t_2) then the remainder term in (19) is of the order $O(\hbar^\infty)$.

3.3. Husimi functions

Let φ_n be an eigenstate of H with the eigenenergy E_n. Given a coherent state ϕ_{z}^σ one can construct the corresponding Husimi function:

$$H_n(\mathsf{z}) = \left|\langle \phi_{\mathsf{z}}^\sigma \,|\, \varphi_n \rangle\right|^2 \qquad \mathsf{z} = (q, p) \qquad \sigma = (\sigma^0, \sigma^1) \qquad -\mathrm{i}\sigma^0 = \beta > 0. \tag{22}$$

Based on the propagation formula (19) the following average over Husimi functions

$$\sum_n f(\omega_n)\left|\langle\varphi_n|\phi_{\mathsf z}^\sigma\rangle\right|^2=\sum_{l=0}^{\infty}d_l\hbar^{\frac12+l}\qquad \omega_n=\frac{E_n-E}{\hbar}\qquad E=p^2/2 \tag{23}$$

has been calculated to the leading order by Paul and Uribe [16]. It turns out that the result depends on whether the classical trajectory through $\mathsf z$ is periodic or not. With the application to the Hamiltonian $\mathsf H_\Omega$ the results in [16] read as follows. Let $\tilde f(\cdot)$ be the Fourier transform of f. If $\mathsf z$ is not periodic under the flow Ψ_Ω^t then

$$d_0=\left(\frac{1}{\beta E}\right)^{1/2}\tilde f(0). \tag{24}$$

Alternatively, if $\mathsf z$ belongs to a periodic trajectory additional terms (of the same order in $\hbar$) arise. In particular, for a hyperbolic periodic trajectory γ with the period T_γ the leading term in (23) is given by

$$d_0=\left(\frac{1}{\beta E}\right)^{1/2}\left(\sum_{l=-\infty}^{+\infty}\tilde f(lT_\gamma)\frac{\mathrm e^{\mathrm i l(S_\gamma/\hbar+\mu_\gamma)}}{\cosh^{1/2}(l\lambda_\gamma)}\right) \tag{25}$$

where $S_\gamma = 2ET_\gamma$, λ_γ are the action, Lyapunov exponent of γ and μ_γ is the sum of the Maslov index and 'boundary' phase. (Strictly speaking, equation (25) has been obtained in [16] for a class of smooth potentials v. However, since the derivation of (25) is essentially based on the propagation formula (19), the generalization of the above result to the billiards is straightforward.)

4. PW approximation for eigenstates of non-convex billiards (elliptic case)

Let γ be a periodic orbit in the billiard Ω and let $\Gamma(E)$ be the 'lift' of γ to the phase space V at the energy E. This means $\Gamma(E)$ is a set of the points $\mathsf z=(q,p)\in V$ such that $q\in\gamma$, $p^2=2E$ and the vector p is directed along γ. Obviously, for any $\mathsf z\in\Gamma(E)$, $\Psi_\Omega^{T_\gamma}\cdot\mathsf z=\mathsf z$, where T_γ is the period of the trajectory. We will make use of the letter ε to denote the restriction of γ, $\Gamma(E)$ to the domain Ω_ε i.e. $\gamma^\varepsilon=\{q\in\gamma\cap\Omega_\varepsilon\}$, $\Gamma^\varepsilon(E)=\{\mathsf z=(q,p)\in\Gamma(E):q\in\Omega_\varepsilon\}$. Provided that γ is elliptic a set of approximate solutions (*quasi-modes*) $\tilde\varphi_n(x)$ of equations (1) and (2) associated with γ can be constructed. The possibility of quasi-mode construction on elliptic periodic orbits is well known. In the following we will follow the approach developed in [16, 17] (see also [18, 20] and the references therein).

Before we turn to the construction of the states $\tilde\varphi_n(x)$ in billiards let us recall a general definition for quasi-modes.

Definition. *Let H be a Hilbert space and $\mathsf H$ be a self-adjoint operator with the domain $D(H)$. A pair $(\tilde\varphi_n,\widetilde E_n)$ with $\tilde\varphi_n\in D(H)$, $\|\tilde\varphi_n\|=1$ and $\widetilde E_n\in\mathbf R$ is called a quasi-mode with the discrepancy δ_n, if*

$$(\mathsf H-\widetilde E_n)\tilde\varphi_n=r_n\qquad \textit{with}\quad \|r_n\|=\delta_n. \tag{26}$$

By a general theory (see, e.g., [19]) the quasi-modes $(\tilde\varphi_n,\widetilde E_n)$ should be close to an exact solution (φ_n,E_n) of the equation

$$(\mathsf H-E)\varphi=0 \tag{27}$$

in the following sense. If $(\tilde\varphi,\widetilde E)$ is a quasi-mode with the discrepancy δ then there exists at least one eigenvalue of $\mathsf H$ in the interval

$$\mathcal P_\delta=[\widetilde E-\delta,\widetilde E+\delta]. \tag{28}$$

Furthermore, let ν be the distance between $\widetilde{E}$ and an eigenvalue E_i of H outside $\mathcal{P}_\delta$, then

$$\|\tilde{\varphi} - \pi_\nu \tilde{\varphi}\| \leqslant \frac{\delta}{\nu} \tag{29}$$

where π_ν denotes the spectral projection operator on the part of the spectrum $\{E_n\}$ inside the interval $(\widetilde{E} - \nu, \widetilde{E} + \nu)$.

Remark. In general, formula (29) implies that any state $\tilde{\varphi}_n$ approximates a superposition of eigenstates φ_n. In order to approximate individual eigenstates of H, δ_n should be much less than the energy intervals: $\Delta E_n = |E_n - E_{n+1}|$, $\Delta E_{n-1} = |E_n - E_{n-1}|$. For two-dimensional billiards $\langle \Delta E_n \rangle \sim \hbar^2$, so the approximation of φ_n by $\tilde{\varphi}_n$ becomes semiclassically ($\hbar \to 0$) meaningful only if the spectrum of Ω has no systematic degeneracies and quasi-modes with discrepancy $\delta \sim \hbar^\alpha, \alpha > 2$ can be constructed. For the quantum billiard problem a quasi-mode construction providing $\delta = O(\hbar^\infty)$ is known to exist [21] and for the rest of this section we will assume that the billiard spectrum has no systematic degeneracies.

4.1. Quasi-mode construction

We will now schematically describe the construction of quasi-modes concentrated on elliptic periodic orbits. The basic idea is to launch a coherent state along the orbit and average over time. As can be shown, this procedure yields an approximately invariant state if the initial state is chosen in the right way, see, e.g., [16, 20]. Let ϕ_{z}^σ, $\mathsf{z} = (q, p) \in \Gamma^\varepsilon(E)$ be a coherent state localized on the periodic orbit γ. We will associate with γ the state

$$\left|\Phi_{\Gamma(E)}^\sigma\right\rangle = \frac{1}{C} \int_0^{T_\gamma} \mathrm{e}^{\mathrm{i}t(E - \mathsf{H}_\Omega)/\hbar} \left|\phi_{\mathsf{z}}^\sigma\right\rangle \mathrm{d}t \tag{30}$$

where C is fixed by the normalization condition $\left\|\Phi_{\Gamma(E)}^\sigma\right\| = 1$ and T_γ is the period of the classical evolution along $\gamma : \mathsf{z}(T_\gamma) = \mathsf{z}$. The propagation formula (19) yields

$$(E - \mathsf{H}_\Omega)\Phi_{\Gamma(E)}^\sigma = r_\gamma \qquad C r_\gamma = \mathrm{i}\hbar\left(\mathrm{e}^{\mathrm{i}(S_\gamma/\hbar + \mu_\gamma)} \phi_{\mathsf{z}}^{\sigma(T_\gamma)} - \phi_{\mathsf{z}}^\sigma\right) + O(\hbar^{3/2}) \tag{31}$$

where S_γ is the classical action and μ_γ is the sum of the Maslov index and the 'boundary' phase after one period. Therefore, $C r_\gamma = O(\hbar^{3/2})$ provided that the following conditions are satisfied: *condition 1*: $\sigma(T_\gamma) = \sigma$; *condition 2:* $S_\gamma/\hbar + \mu_\gamma = 2\pi n$ for some integer n.

For each n let $\mathcal{E}_n, \sigma_n = \left(\sigma_n^0, \sigma_n^1\right)$ denote solutions of conditions (1) and (2). It is possible to show (see, e.g., [16]) that the first condition can be satisfied if and only if $\sigma_n^0 = 0$ and γ is an elliptic periodic orbit. The second condition imposes the Bohr–Sommerfeld quantization on the quasi-energy $\mathcal{E}_n$. When both conditions are satisfied the corresponding pair $\left(\mathcal{E}_n, \Phi_{\Gamma(\mathcal{E}_n)}^{\sigma_n}\right)$ provides the quasi-mode with the discrepancy $\delta_\gamma = O(\hbar^{3/2})/C$.

Remark. It should be noted that a much wider class of quasi-modes concentrated on γ can be constructed by this method if one uses in (30) coherent states with transverse excitations [16, 18]. For simplicity of exposition, we restrict our consideration only to the quasi-modes without transverse excitations, whose leading order is determined by equation (30).

To construct quasi-modes with discrepancies of higher order in $\hbar$ one has to consider the time evolution of coherent states of a more general type. This leads to transport equations whose solvability poses additional conditions on the quasi-energies, see [20]. From the results of Cardoso and Popov [21], the possibility of constructing quasi-modes $(\widetilde{E}_n, \tilde{\varphi}_n)$ in billiards having discrepancy $\delta_\gamma = O(\hbar^\infty)$ is known to exist. Let (s, y) be a coordinate system in a neighbourhood of γ such that s is a coordinate along the trajectory and y is a coordinate in the orthogonal direction. Using these coordinates the leading order of $(\widetilde{E}_n, \tilde{\varphi}_n)$ can be written as follows [15, 20]:

$$\widetilde{E}_n = \mathcal{E}_n + O(\hbar^2) \qquad \tilde{\varphi}_n(x) = \mathrm{e}^{\mathrm{i}v(x)/\hbar}u(x) + O(\hbar) \tag{32}$$

where

$$v(s, y) = v_0(s)y^2 + O(y^3) \qquad u(s, y) = u_0(s) + O(y^2)$$

and the parameters $v_0(s)$, $u_0(s)$ are determined by conditions (1) and (2):

$$\Phi^{\sigma_n}_{\Gamma(\mathcal{E}_n)}(x) = \mathrm{e}^{\mathrm{i}v_0(s)y^2/\hbar}u_0(s) \qquad x = (s, y). \tag{33}$$

As has been explained before, in the absence of systematic degeneracies in the billiard spectrum one can expect that, in general, a state $\tilde{\varphi}_n$ approximates an individual eigenstate of the billiard Ω. In what follows, we will denote by $\widetilde{\mathcal{S}}_\gamma$ the set of quasi-modes for which $\tilde{\varphi}_n$ approximates some eigenstate φ_n (rather than a linear combination of φ_n) and by $\mathcal{S}_\gamma$ the set of true solutions of equations (1) and (2) corresponding to $\widetilde{\mathcal{S}}_\gamma$. Then from equation (29) for each $(\tilde{\varphi}_i, \widetilde{E}_i) \in \widetilde{\mathcal{S}}_\gamma$ and $(\varphi_i, E_i) \in \mathcal{S}_\gamma$ we have

$$\mathcal{C}^1_i = \|\tilde{\varphi}_i - \varphi_i\| = O(\hbar^\infty) \qquad |\widetilde{E}_i - E_i| = O(\hbar^\infty). \tag{34}$$

4.2. A lower bound for the approximation of eigenstates

The quasi-mode construction described in the previous section is quite general and can be applied to an arbitrary elliptic periodic trajectory. In the present section we will consider eigenstates of the billiard Ω from the subset $\mathcal{S}_\gamma$, where γ is an elliptic SPT. We show that for $(\varphi_n, E_n) \in \mathcal{S}_\gamma$ and any regular solution $\psi \in \mathcal{M}(E_n)$ of equation (1) in $\mathbf{R}^2$ the norm

$$\eta_n(\psi) = \|\varphi_n - \psi\| \tag{35}$$

is bounded from below by

$$\eta_n(\psi) \geqslant \mathcal{C}_\gamma + \mathcal{C}^1_n + \mathcal{O}'_1 \tag{36}$$

where $\mathcal{O}'_1 = O(\hbar^{1/2})$ and $\mathcal{C}_\gamma$ is a positive constant determined only by geometrical parameters of the periodic orbit. Since $\mathcal{C}^1_n = O(\hbar^\infty)$, this implies that the inequality (14) holds for any $(\varphi_n, E_n) \in \mathcal{S}_\gamma$.

Let γ be an elliptic SPT and let $\gamma_1, \bar{\gamma}_1$ be as defined in section 2, see figure 3. Now fix the parameter ε to be sufficiently small such that $\gamma^\varepsilon_1 \equiv \gamma_1 \cap \Omega_\varepsilon \neq \emptyset$, $\bar{\gamma}^\varepsilon_1 \equiv \bar{\gamma}_1 \cap \Omega_\varepsilon \neq \emptyset$. We will denote by the capital letters $\Gamma_1(E), \bar{\Gamma}_1(E)$ (respectively, $\Gamma^\varepsilon_1(E), \bar{\Gamma}^\varepsilon_1(E)$) the corresponding 'lifts' of $\gamma_1, \bar{\gamma}_1$ (respectively $\gamma^\varepsilon_1, \bar{\gamma}^\varepsilon_1$) into the phase space V at the energy shell E. Recall that the main idea behind the quasi-mode construction (30) is to use coherent states propagating along a periodic orbit. By analogy, one can construct states localized on γ_1 and $\bar{\gamma}_1$. Let $\mathsf{z}(0) = \mathsf{z} \in \Gamma_1(E)$. Consider the classical evolution (both for positive and negative time) of z under the free flow $\Psi^t_0 : \mathsf{z} \to \mathsf{z}(t) = (q(t), p(t))$ in $\mathbf{R}^2$. Obviously, as time evolves, the point $q(t)$ successively crosses the boundary of Ω_ε at the sequence of points $q_1, q_2, \bar{q}_1, \bar{q}_2$, see figure 3. We will denote by $t_1, t_2, \bar{t}_1, \bar{t}_2$ the corresponding time moments: $q_1 = q(t_1), q_2 = q(t_2), \bar{q}_1 = q(\bar{t}_1), \bar{q}_2 = q(\bar{t}_2)$. Then the states localized along γ_1 and $\bar{\gamma}_1$ are given by

$$\left|\Phi^\sigma_{\Gamma_1(E)}\right\rangle = \frac{1}{T_{\gamma_1}}\int_{t_2}^{t_1} \mathrm{e}^{\mathrm{i}t(E-\mathsf{H}_0)/\hbar}\left|\phi^\sigma_{\mathsf{z}}\right\rangle \mathrm{d}t \qquad T_{\gamma_1} = |t_1 - t_2| \tag{37}$$

$$\left|\Phi^\sigma_{\bar{\Gamma}_1(E)}\right\rangle = \frac{1}{T_{\bar{\gamma}_1}}\int_{\bar{t}_2}^{\bar{t}_1} \mathrm{e}^{\mathrm{i}t(E-\mathsf{H}_0)/\hbar}\left|\phi^\sigma_{\mathsf{z}}\right\rangle \mathrm{d}t \qquad T_{\bar{\gamma}_1} = |\bar{t}_1 - \bar{t}_2|. \tag{38}$$

Note, that under the free evolution $\mathrm{e}^{-\mathrm{i}t\mathsf{H}_0/\hbar}$ the support of $\phi^{\sigma}_{\mathsf{z}}$ is not preserved inside Ω, and therefore the supports of $\Phi^{\sigma}_{\Gamma_1}, \Phi^{\sigma}_{\bar{\Gamma}_1}$ do not belong to the billiard domain. However, one can slightly modify the definition of the states $\Phi^{\sigma}_{\Gamma_1}, \Phi^{\sigma}_{\bar{\Gamma}_1}$ to make them admissible as billiard states in Ω. Let $\mathsf{z} = \mathsf{z}_1, \sigma = \sigma_1$ be as before and set τ such that under the classical evolution $\Psi^{\tau}_0 : \mathsf{z}_1 \to \mathsf{z}(\tau)$ the point $\mathsf{z}(\tau) = \mathsf{z}_2$ belongs to $\bar{\Gamma}^{\varepsilon}_1$. Set $\phi^{\sigma_2}_{\mathsf{z}_2}(x) = \mathrm{e}^{-\mathrm{i}\tau\mathsf{H}_0/\hbar}\phi^{\sigma}_{\mathsf{z}}(x) + O(\hbar^{\infty})$ as the coherent state in Ω, whose parameters are given by $(\sigma_2, \mathsf{z}_2) = \left(D\Psi^{\tau}_0 \cdot \sigma_1, \Psi^{\tau}_0 \cdot \mathsf{z}_1\right)$. Then the states

$$\left|\bar{\Phi}^{\sigma}_{\Gamma_1(E)}\right\rangle = \frac{1}{T_{\gamma_1}} \int_{t_2}^{t_1} \mathrm{e}^{\mathrm{i}t(E-\mathsf{H}_\Omega)/\hbar} \left|\phi^{\sigma_1}_{\mathsf{z}_1}\right\rangle \mathrm{d}t \tag{39}$$

$$\left|\bar{\Phi}^{\sigma}_{\bar{\Gamma}_1(E)}\right\rangle = \frac{1}{T_{\bar{\gamma}_1}} \int_{\bar{t}_2-\tau}^{\bar{t}_1-\tau} \mathrm{e}^{\mathrm{i}t(E-\mathsf{H}_\Omega)/\hbar} \left|\phi^{\sigma_2}_{\mathsf{z}_2}\right\rangle \mathrm{d}t \tag{40}$$

have their supports in Ω and satisfy

$$\left|\bar{\Phi}^{\sigma}_{\Gamma_1(E)}\right\rangle = \left|\Phi^{\sigma}_{\Gamma_1(E)}\right\rangle + O(\hbar^{\infty}) \qquad \left|\bar{\Phi}^{\sigma}_{\bar{\Gamma}_1(E)}\right\rangle = \left|\Phi^{\sigma}_{\bar{\Gamma}_1(E)}\right\rangle + O(\hbar^{\infty}). \tag{41}$$

To get the lower bound (36) we are going to first construct a vector Φ with the property

$$\langle\psi|\Phi\rangle = 0 + O(\hbar^{\infty}) \tag{42}$$

for any $\psi \in \mathcal{M}(E')$. Let us show how such a vector can be constructed using $\bar{\Phi}^{\sigma}_{\Gamma_1}, \bar{\Phi}^{\sigma}_{\bar{\Gamma}_1}$. Set $\langle\cdot|\cdot\rangle_{\mathbf{R}^2}, \langle\cdot|\cdot\rangle$ as the scalar products in $L^2(\mathbf{R}^2)$ and $L^2(\Omega)$, respectively. From definitions (37) and (38) one has

$$\left\langle\psi\middle|\Phi^{\sigma}_{\Gamma_1(E)}\right\rangle_{\mathbf{R}^2} = \frac{1}{T_{\gamma_1}} \int_{t_2}^{t_1} \mathrm{e}^{\mathrm{i}t(E-E')/\hbar}\left\langle\psi\middle|\phi^{\sigma}_{\mathsf{z}}\right\rangle \mathrm{d}t = C_1(\omega)\left\langle\psi\middle|\phi^{\sigma}_{\mathsf{z}}\right\rangle \tag{43}$$

where

$$C_1(\omega) = \exp\left(\frac{\mathrm{i}(t_1+t_2)\omega}{2}\right)\frac{2\sin(\omega T_{\gamma_1}/2)}{\omega T_{\gamma_1}} \tag{44}$$

and $\omega = (E - E')/\hbar$. Analogously:

$$\left\langle\psi\middle|\Phi^{\sigma}_{\bar{\Gamma}_1(E)}\right\rangle_{\mathbf{R}^2} = C_2(\omega)\left\langle\psi\middle|\phi^{\sigma}_{\mathsf{z}}\right\rangle \tag{45}$$

with

$$C_2(\omega) = \exp\left(\frac{\mathrm{i}(\bar{t}_1+\bar{t}_2)\omega}{2}\right)\frac{2\sin\left(\omega T_{\bar{\gamma}_1}/2\right)}{\omega T_{\bar{\gamma}_1}}. \tag{46}$$

Furthermore, let us introduce the states

$$\left|\Phi^{\sigma}_1(E,E')\right\rangle = \frac{1}{C_1(\omega)}\left|\bar{\Phi}^{\sigma}_{\Gamma_1(E)}\right\rangle \qquad \left|\Phi^{\sigma}_2(E,E')\right\rangle = \frac{1}{C_2(\omega)}\left|\bar{\Phi}^{\sigma}_{\bar{\Gamma}_1(E)}\right\rangle. \tag{47}$$

Then it follows immediately from equations (43) and (45) that the vector $\Phi = \Phi^{\sigma}(E, E')$,

$$|\Phi^{\sigma}(E,E')\rangle = \left|\Phi^{\sigma}_1(E,E')\right\rangle - \left|\Phi^{\sigma}_2(E,E')\right\rangle \tag{48}$$

satisfies orthogonality condition (42).

Let $(\varphi_n, E_n) \in \mathcal{S}_\gamma$ be a solution of equations (1) and (2) and let $(\tilde{\varphi}_n, \tilde{E}_n) \in \widetilde{\mathcal{S}}_\gamma$ be the corresponding quasi-mode, whose leading order parameters $\mathcal{E}_n, \sigma_n = \left(\sigma^0_n, \sigma^1_n\right)$ are determined by conditions (1) and (2), see equations (32) and (33). Now fix the energy parameters in equation (48) by $E = \mathcal{E}_n$, $E' = E_n$ and put $\sigma = \bar{\sigma}_n$, where $\bar{\sigma}_n = \left(\mathrm{i}\beta, \sigma^1_n\right)$ and β is an arbitrary real positive number. We will make use of the vector

$$|\Phi_n\rangle = |\Phi^{\bar{\sigma}_n}(\mathcal{E}_n, E_n)\rangle$$

in order to get a lower bound on η_n. For any $\psi \in \mathcal{M}(E_n)$ we have

$$\|\tilde{\varphi}_n - \psi\|\|\Phi_n\| \geqslant |\langle\tilde{\varphi}_n - \psi|\Phi_n\rangle| = |\langle\tilde{\varphi}_n|\Phi_n\rangle| + O(\hbar^\infty). \tag{49}$$

Using the triangle inequality

$$\|\tilde{\varphi}_n - \varphi_n\| + \|\varphi_n - \psi\| \geqslant \|\tilde{\varphi}_n - \varphi_n + \varphi_n - \psi\| = \|\tilde{\varphi}_n - \psi\| \tag{50}$$

one gets immediately from (49)

$$\eta_n(\psi) = \|\varphi_n - \psi\| \geqslant \frac{|\langle\tilde{\varphi}_n|\Phi_n\rangle|}{\|\Phi_n\|} - \mathcal{C}_n^1 + \mathcal{O}_2', \mathcal{O}_2' = O(\hbar^\infty). \tag{51}$$

It remains to estimate the scalar product $|\langle\tilde{\varphi}_n|\Phi_n\rangle|$ and the norm of the vector Φ_n (note that by definition, Φ_n is not normalized). First, consider the norm $\|\Phi_n\|$. Since $\gamma_1 \cap \bar{\gamma}_1 = \emptyset$ one has from the definition of Φ_n

$$\langle\Phi_n|\Phi_n\rangle = \frac{1}{|C_1(\omega_n)|^2}\left\langle\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\right\rangle + \frac{1}{|C_2(\omega_n)|^2}\left\langle\Phi^{\bar{\sigma}_n}_{\bar{\Gamma}_1(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\bar{\Gamma}_1(\mathcal{E}_n)}\right\rangle + O(\hbar^\infty) \tag{52}$$

with $\omega_n = (E_n - \mathcal{E}_n)/\hbar$. The calculations of the scalar products performed in the appendix give

$$\begin{aligned} \left\langle\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\right\rangle &= \frac{1}{T_{\gamma_1}}\left(\frac{2\pi\hbar}{\beta E_n}\right)^{1/2} + O(\hbar) \\ \left\langle\Phi^{\bar{\sigma}_n}_{\bar{\Gamma}_1(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\bar{\Gamma}_1(\mathcal{E}_n)}\right\rangle &= \frac{1}{T_{\bar{\gamma}_1}}\left(\frac{2\pi\hbar}{\beta E_n}\right)^{1/2} + O(\hbar) \end{aligned} \tag{53}$$

and for the leading order of $C_1(\omega_n)$, $C_2(\omega_n)$ one has from equations (44) and (46)

$$|C_1(\omega_n)| = 1 + O(\hbar) \qquad |C_2(\omega_n)| = 1 + O(\hbar). \tag{54}$$

Combining (53) and (54) together one finally gets

$$\langle\Phi_n|\Phi_n\rangle = \left(\frac{2\pi\hbar}{\beta E_n}\right)^{1/2}\left(\frac{1}{T_{\gamma_1}} + \frac{1}{T_{\bar{\gamma}_1}}\right) + O(\hbar). \tag{55}$$

In the same way for the scalar product $\langle\tilde{\varphi}_n|\Phi_n\rangle$ we have from (32) and (33)

$$\begin{aligned} |\langle\tilde{\varphi}_n|\Phi_n\rangle| &= \left|\left\langle\Phi^{\sigma_n}_{\Gamma(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\right\rangle\right| + O(\hbar) = \frac{T_{\gamma_1}}{T_\gamma}\left|\left\langle\Phi^{\sigma_n}_{\Gamma(\mathcal{E}_n)}\middle|\Phi^{\sigma_n}_{\Gamma(\mathcal{E}_n)}\right\rangle\right|^{1/2}\left|\left\langle\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\middle|\Phi^{\bar{\sigma}_n}_{\Gamma_1(\mathcal{E}_n)}\right\rangle\right|^{1/2} + O(\hbar) \\ &= \frac{1}{T_\gamma}\left(\frac{2\pi\hbar}{\beta E_n}\right)^{1/2} + O(\hbar). \end{aligned} \tag{56}$$

The estimation (36) follows now immediately after inserting equations (55) and (56) into (51). The resulting constant $\mathcal{C}_\gamma$, which determines the lower bound on η_n in the semiclassical limit reads

$$\mathcal{C}_\gamma = \sqrt{\frac{T_{\bar{\gamma}_1}T_{\gamma_1}}{(T_{\bar{\gamma}_1} + T_{\gamma_1})T_\gamma}} = \sqrt{\frac{\ell_{\bar{\gamma}_1}\ell_{\gamma_1}}{(\ell_{\bar{\gamma}_1} + \ell_{\gamma_1})\ell_\gamma}} + O(\varepsilon) \tag{57}$$

where $\ell_{\bar{\gamma}_1}, \ell_{\gamma_1}, \ell_\gamma$ are the lengths of $\bar{\gamma}_1, \gamma_1$ and γ, respectively.

5. PW approximation for eigenstates of non-convex billiards (hyperbolic case)

In the present section we consider the case of a hyperbolic SPT γ. As before, let E_n be the nth eigenenergy of the billiard problem (1) and (2) and let $\varphi_n(x)$ be the corresponding eigenfunction approximated by a regular solution $\psi_n(x) \in \mathcal{M}(E_n)$ of equation (1). For an arbitrary set of $\psi_n(x) \in \mathcal{M}(E_n)$, $n = 1, 2, \ldots, \infty$ we will estimate the average of

$$\eta_n = \|\varphi_n - \psi_n\| \tag{58}$$

over an energy interval. Our objective is to show that independent of the choice of ψ_n, in the limit $\hbar \to 0$ the average $\langle \eta_i \rangle$ is bounded from below by a strictly positive constant.

Let $\Phi_1^\sigma(E, E')$, $\Phi_2^\sigma(E, E')$, $\Phi^\sigma(E, E')$ be as in the previous section with the parameter σ of the form $\sigma = (\mathrm{i}\beta, \sigma^1)$, $\beta > 0$. For each integer n we will consider the states

$$|\Phi_{n,1}\rangle = \left|\Phi_1^\sigma(E, E_n)\right\rangle \qquad |\Phi_{n,2}\rangle = \left|\Phi_2^\sigma(E, E_n)\right\rangle \tag{59}$$

and their difference

$$|\widetilde{\Phi}_n\rangle = |\Phi_{n,1}\rangle - |\Phi_{n,2}\rangle = |\Phi^\sigma(E, E_n)\rangle \tag{60}$$

which is orthogonal to any $\psi \in \mathcal{M}(E_n)$ up to the term $O(\hbar^\infty)$ (see equation (42)). In addition, it will also be useful to introduce the vector

$$|\widetilde{\Phi}'_n\rangle = |\Phi_{n,1}\rangle + |\Phi_{n,2}\rangle. \tag{61}$$

Note that $\widetilde{\Phi}'_n$ is orthogonal to $\widetilde{\Phi}_n$ in the semiclassical limit.

Similarly to the case of elliptic SPT, one can make use of the vector $\widetilde{\Phi}_n$ to get a lower bound on η_n:

$$\eta_n \geqslant \frac{|\langle \widetilde{\Phi}_n | \varphi_n - \psi_n \rangle|}{\|\widetilde{\Phi}_n\|} = \frac{|\langle \widetilde{\Phi}_n | \varphi_n \rangle|}{\|\widetilde{\Phi}_n\|} + O(\hbar^\infty). \tag{62}$$

In order to estimate the right-hand side of this inequality let us consider the following difference:

$$\mathcal{D}_n = |\langle \Phi_{n,1} | \varphi_n \rangle|^2 - |\langle \Phi_{n,2} | \varphi_n \rangle|^2. \tag{63}$$

Using the vectors $\widetilde{\Phi}_n$, $\widetilde{\Phi}'_n$ one can rewrite $\mathcal{D}_n$ as

$$\mathcal{D}_n = \mathrm{Re}(\langle \widetilde{\Phi}_n | \varphi_n \rangle \langle \widetilde{\Phi}'_n | \varphi_n \rangle^*). \tag{64}$$

Hence, the following inequality follows immediately:

$$|\mathcal{D}_n| \leqslant |\langle \widetilde{\Phi}_n | \varphi_n \rangle| |\langle \widetilde{\Phi}'_n | \varphi_n \rangle| \leqslant \|\widetilde{\Phi}'_n\| |\langle \widetilde{\Phi}_n | \varphi_n \rangle|. \tag{65}$$

Finally, since $\|\widetilde{\Phi}_n\| - \|\widetilde{\Phi}'_n\| = O(\hbar^\infty)$, we get from (62) and (65)

$$\eta_n \geqslant \frac{|\mathcal{D}_n|}{\|\widetilde{\Phi}_n\| \|\widetilde{\Phi}'_n\|} + \mathcal{O}_1 = \left| \frac{|\langle \Phi_{n,1} | \varphi_n \rangle|^2 - |\langle \Phi_{n,2} | \varphi_n \rangle|^2}{\langle \widetilde{\Phi}_n | \widetilde{\Phi}_n \rangle} \right| + \mathcal{O}_2 \tag{66}$$

where the terms $\mathcal{O}_1$, $\mathcal{O}_2$ are of order $O(\hbar^\infty)$.

We will now use this inequality to get a lower bound for the sum of η_n over the energy interval $\mathcal{P}_{c\hbar} = [E - c\hbar, E + c\hbar]$, where c is a positive constant. One has straightforwardly from (66)

$$\sum_{E_n \in \mathcal{P}_{c\hbar}} \eta_n > \left| \sum_{E_n \in \mathcal{P}_{c\hbar}} \frac{|\langle \Phi_{n,1} | \varphi_n \rangle|^2}{\langle \widetilde{\Phi}_n | \widetilde{\Phi}_n \rangle} - \sum_{E_n \in \mathcal{P}_{c\hbar}} \frac{|\langle \Phi_{n,2} | \varphi_n \rangle|^2}{\langle \widetilde{\Phi}_n | \widetilde{\Phi}_n \rangle} \right| + \mathcal{O}_3 \qquad \mathcal{O}_3 = O(\hbar^\infty). \tag{67}$$

Furthermore, the definition of the states $\Phi_{n,1}$, $\Phi_{n,2}$ implies

$$|\langle\Phi_{n,1}|\varphi_n\rangle|^2 = \left|\langle\phi^{\sigma_1}_{\mathsf{z}_1}|\varphi_n\rangle\right|^2 \qquad \mathsf{z}_1 \in \Gamma^\varepsilon_1 \qquad |\langle\Phi_{n,2}|\varphi_n\rangle|^2 = \left|\langle\phi^{\sigma_2}_{\mathsf{z}_2}|\varphi_n\rangle\right|^2 \qquad \mathsf{z}_2 \in \bar{\Gamma}^\varepsilon_1 \tag{68}$$

where $(\mathsf{z}_1, \sigma_1) = (\mathsf{z}, \sigma)$ and $(\mathsf{z}_2, \sigma_2) = (\mathsf{z}(\tau), \sigma(\tau))$ are related by the free classical evolution as in the previous section. As a result, the inequality (67) reads

$$\sum_{E_n\in\mathcal{P}_{c\hbar}} \eta_n > \left| \sum_n f(\omega_n) \left|\langle\varphi_n|\phi^{\sigma_1}_{\mathsf{z}_1}\rangle\right|^2 - \sum_n f(\omega_n) \left|\langle\varphi_n|\phi^{\sigma_2}_{\mathsf{z}_2}\rangle\right|^2 \right| + \mathcal{O}_3 \tag{69}$$

with $\omega_n = (E - E_n)/\hbar$ and

$$f(\omega_n) = \begin{cases} 1/\langle\widetilde{\Phi}_n|\widetilde{\Phi}_n\rangle & \text{if} \quad \omega_n \in [-c, c] \\ 0 & \text{otherwise.} \end{cases}$$

The elementary calculations (see the appendix) provide the leading order of the function $f(\omega_n)$, $\omega_n \in [-c, c]$:

$$\begin{aligned} f(\omega_n) &= \frac{1}{\langle\Phi_{n,1}|\Phi_{n,1}\rangle + \langle\Phi_{n,2}|\Phi_{n,2}\rangle} + O(\hbar^\infty) \\ &= \frac{2|p|}{(\pi\hbar\beta)^{\frac{1}{2}}} \left(\frac{\omega_n^2 T_{\gamma_1}}{\sin^2(\omega_n T_{\gamma_1}/2)} + \frac{\omega_n^2 T_{\bar{\gamma}_1}}{\sin^2(\omega_n T_{\bar{\gamma}_1}/2)} \right)^{-1} + O(\hbar^0). \end{aligned} \tag{70}$$

Now we can apply to (69) the results of Paul and Uribe (see section 3). Taking into account that $\mathsf{z}_1 \in \Gamma$ while z_2 does not belong to any periodic trajectory, we get from equations (24) and (25) the following estimation for the average of η_n:

$$\langle\eta_i\rangle \equiv \frac{1}{\#\mathcal{P}_{c\hbar}} \sum_{E_n\in\mathcal{P}_{c\hbar}} \eta_n > \frac{1}{\#\mathcal{P}_{c\hbar}} \left| \sum_{l\neq 0} \widetilde{F}(lT_\gamma) \frac{\mathrm{e}^{\mathrm{i}l(S_\gamma/\hbar+\mu_\gamma)}}{\cosh^{1/2}(l\lambda_\gamma)} \right| + \mathcal{O}_4 \tag{71}$$

where $\mathcal{O}_4 = O(\hbar^{3/2})$, $\widetilde{F}(\cdot)$ is the Fourier transform of the function

$$F(x) = \begin{cases} \left(\frac{8}{\pi}\right)^{\frac{1}{2}} \left(\frac{x^2 T_{\gamma_1}}{\sin^2(xT_{\gamma_1}/2)} + \frac{x^2 T_{\bar{\gamma}_1}}{\sin^2(xT_{\bar{\gamma}_1}/2)} \right)^{-1} & \text{if} \quad x \in [-c, c] \\ 0 & \text{otherwise} \end{cases}$$

and $\#\mathcal{P}_{c\hbar}$ is the number of eigenstates in the interval $\mathcal{P}_{c\hbar}$ whose leading order for a billiard of area $\mathcal{A}$ is given by the Weyl formula:

$$\#\mathcal{P}_{c\hbar} = \mathcal{A}c/2\pi\hbar + O(\hbar^0).$$

Consequently, if

$$Y = \left| \sum_{l\neq 0} \widetilde{F}(lT_\gamma) \frac{\mathrm{e}^{\mathrm{i}l(S_\gamma/\hbar+\mu_\gamma)}}{\cosh^{1/2}(l\lambda_\gamma)} \right| \neq 0 \tag{72}$$

one has from (71)

$$\langle\eta_i\rangle > \mathcal{B}\hbar + \mathcal{O} \tag{73}$$

where

$$\mathcal{O} = O(\hbar^{3/2}) \qquad \mathcal{B} = 2\pi Y/c\mathcal{A} > 0.$$

If moreover one assumes that $T_{\bar{\gamma}_1}c$, $T_{\gamma_1}c \ll 1$, the function $F(x)$ takes a simple form:

$$F(x) \approx \begin{cases} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left(\frac{T_{\bar{\gamma}_1}T_{\gamma_1}}{T_{\bar{\gamma}_1}+T_{\gamma_1}} \right) & \text{if} \quad x \in [-c, c] \\ 0 & \text{otherwise} \end{cases}$$

and the constant $\mathcal{B}$ can be written explicitly as

$$\mathcal{B} \approx \frac{\sqrt{2\pi}}{\mathcal{A}} \left(\frac{T_{\bar{\gamma}_1} T_{\gamma_1}}{T_{\bar{\gamma}_1} + T_{\gamma_1}} \right) \left| \sum_{l \neq 0} \frac{\sin(lcT_\gamma)}{lcT_\gamma} \frac{\mathrm{e}^{\mathrm{i}l(S_\gamma/\hbar + \mu_\gamma)}}{\cosh^{1/2}(l\lambda_\gamma)} \right|. \tag{74}$$

Note, that the lower bound (73) has been obtained using only one SPT. In the case of hyperbolic dynamics, however, the periodic orbits (and, in particular, SPT) proliferate exponentially. Therefore, one can improve the estimation (73) making use of a vector $\widetilde{\Phi}_n^{\mathrm{sum}}$ which is concentrated on a set of SPT $\{\gamma\}$ and satisfies equation (42). A simple way to construct such a vector is to define it as the superposition:

$$\widetilde{\Phi}_n^{\mathrm{sum}} = \sum_{\{\gamma\}} \widetilde{\Phi}_n(\gamma) \tag{75}$$

where $\widetilde{\Phi}_n(\gamma)$ stands for the vector (60) associated with a SPT γ.

Finally, let us mention that the statistical estimation (73) can be straightforwardly generalized to the case of elliptic SPT. In that case one should use the analogues of equations (24) and (25) (which are known to exist [16]) for stable periodic trajectories.

6. Discussion and conclusions

Speaking informally, proposition 2 implies that there is no on-shell basis of regular solutions of the Helmholtz equation which can be used to approximate all eigenstates of a generic non-convex billiard. This means any linear combination of plane waves, radial waves etc. with the same energy fails to approximate real eigenstates of non-convex billiards. In fact, a stronger result can be shown. Let Ω be a generic non-convex billiard and let Ω' be a domain (not necessarily convex) which properly contains Ω: $\Omega' \supset \Omega, \partial\Omega' \cap \partial\Omega = \emptyset$. Denote by $\mathcal{M}_{\Omega'}(E)$ the set of all solutions of equation (1) regular in Ω' (note, that $\mathcal{M}_{\Omega'}(E) \supseteq \mathcal{M}(E)$). Let us argue that the eigenstates of Ω cannot be approximated, in general, by states belonging to $\mathcal{M}_{\Omega'}(E)$. Let γ be a SPT and let $l, \gamma_1, \bar{\gamma}_1$ be as defined before. Furthermore, assume that the segment of the line l between γ_1 and $\bar{\gamma}_1$ is entirely in Ω', see figure 4. (It seems to be a natural assumption that in a generic case one can always find such a SPT, provided Ω' properly contains Ω). Then take $\Omega_0 \subset \Omega$ to be a convex domain satisfying: $\Omega_0 \cap \gamma_1 \neq \emptyset$, $\Omega_0 \cap \bar{\gamma}_1 \neq \emptyset$. Now, suppose an eigenstate φ_n of Ω can be approximated by states $\psi'(x)$ from $\mathcal{M}_{\Omega'}(E_n)$. According to proposition 1 $\psi'(x)$ can be approximated in Ω_0 by regular solutions of equation (1) and thus for any $\epsilon > 0$ there exists $\psi_\epsilon \in \mathcal{M}(E_n)$ such that $\|\varphi_n(x) - \psi_\epsilon(x)\|_{L^2(\Omega_0)} < \epsilon$. Therefore, applying the same arguments as in section 2 we get

$$H_{\varphi_n}(\mathsf{z}_1) - H_{\varphi_n}(\mathsf{z}_2) = \lim_{\epsilon\to 0} |\langle \mathsf{z}_1|\psi_\epsilon\rangle|^2 - \lim_{\epsilon\to 0} |\langle \mathsf{z}_2|\psi_\epsilon\rangle|^2 = O(\hbar^\infty)$$

where $\mathsf{z}_1 = (q_1, p) \in \Gamma_1(E_n), q_1 \in \gamma_1 \cap \Omega_0$ and $\mathsf{z}_2 = (q_2, p) \in \bar{\Gamma}_1(E_n), q_2 \in \bar{\gamma}_1 \cap \Omega_0$. However, as has been pointed out before, this cannot be true for each n since $\mathsf{z}_2 \notin \Gamma$.

The two properties of generic non-convex billiards follow immediately from the above analysis. First, it is not possible to approximate eigenstates of a generic non-convex billiard Ω also if one includes in the basis $\{\psi^{(n)}(\mathrm{k})\}$ singular solutions of equation (1), e.g., the Neumann parts of radial waves

$$\{Y_n(k|x - x_i|)\, \mathrm{e}^{\mathrm{i}n\theta(x-x_i)}, n \in \mathbb{N}\}$$

with a finite number of singularity points $x_i \in \mathbf{R}^2$. Second, there exists an infinite sequence of eigenstates which do not admit extension into any large domain Ω' properly containing Ω. That means the continuation of the interior eigenstates of a generic non-convex billiard into the exterior domain should be (in general) impossible because of singularities which occur arbitrarily close to the billiard's boundary. The exact nature of such singularities remains an

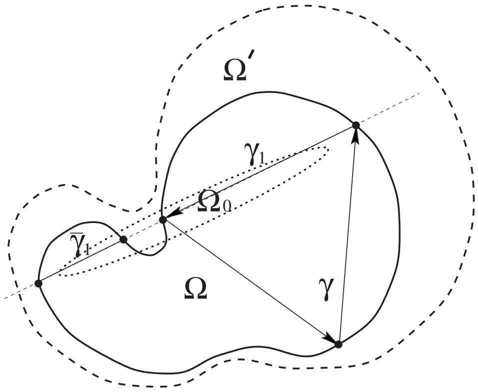

Figure 4. Illustration of the arguments above. If an eigenfunction φ_n can be extended to the exterior domain Ω' (the dashed line) then the PW approximation holds for φ_n in the domain Ω_0 (the dotted line).

open problem. (For example, whether one can, in principle, extend eigenstates beyond the boundary of a generic non-convex billiard.) It should also be mentioned that the problem of the eigenstates extension in convex billiards is beyond the scope of the present paper. It would become a natural question to inquire about the relation between the billiard shape and the type of singularities arising for the extended eigenstates. In particular, it would be interesting to know whether the strong form of spectral duality (when it is possible to extend eigenstates in $\mathbf{R}^2$ as regular solutions of the Helmholtz equation) holds exclusively for integrable billiards.

Further, let us stress an important difference between the cases of elliptic and hyperbolic dynamics. The counting function $\mathcal{N}^*(\mathtt{k}) = \#\{\tilde{\mathtt{k}}_n < \mathtt{k}\}$ for quasi-modes $(\tilde{\varphi}_n, \tilde{\mathtt{k}}_n)$ which can be constructed on an elliptic periodic trajectory is known to be of the same asymptotic form $\mathcal{N}^*(\mathtt{k}) = \alpha \mathtt{k}^2 + O(\mathtt{k})$, $\alpha > 0$ as the counting function $\mathcal{N} = \mathcal{A}\mathtt{k}^2/4\pi + O(\mathtt{k})$ for the real spectrum $\{\mathtt{k}_n\}$, see [21]. Therefore, in a generic case, if an elliptic SPT γ exists the subsequence $\{\varphi_{j_n}, n \in \mathbb{N}\}$ of billiard eigenstates approximated by the quasi-modes concentrated on γ should be of positive density:

$$\lim_{N\to\infty} \frac{1}{N} \#\{j_n | j_n \leqslant N\} = \lim_{\mathtt{k}\to\infty} \frac{\mathcal{N}^*(\mathtt{k})}{\mathcal{N}(\mathtt{k})} > 0.$$

Since for each φ_{j_n} the estimation (14) holds, that means there exists a subsequence of eigenstates with a positive density which do not admit approximation by plane waves. In the case of hyperbolic dynamics the statistical lower bound (13) implies, in fact, only a weaker result. It says that an infinite sequence (possibly of zero density) of such states exists. However, if one assumes that all eigenstates of fully chaotic billiards have ‘uniform properties’ the inequality (13) suggests a natural conjecture:

Conjecture. *For a non-convex billiard with fully chaotic dynamics the set of states which can be approximated by PW is of density zero.*

Note, that it is impossible to exclude the possibility of existence of ‘exceptional’ eigenstates (the eigenstates which can be approximated by PW) in non-convex billiards. Indeed, one can take a finite superposition of plane waves $\psi^{[N]}$ (see equation (5)) and set a (non-convex) nodal domain of $\psi^{[N]}$ to be the billiard’s boundary. Then the corresponding billiard has (at least) one eigenstate $\psi^{[N]}$ which can be approximated by PW.

Finally, the study of the present paper is restricted to the two-dimensional simply connected domains with Dirichlet boundary conditions. However, it is easy to see that the

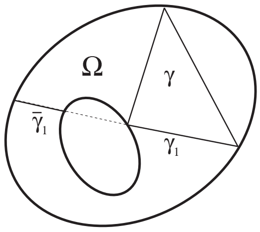

Figure 5. Billiard in a multiply connected domain Ω and a typical SPT γ.

presented results allow several rather straightforward generalizations. First, higher dimensional billiards and different types of boundary conditions can be treated in the same way. Second, billiards in multiply connected domains (figure 5) have the same properties as non-convex billiards. Consequently, all the results obtained for non-convex billiards hold for multiply connected billiards as well. Third, we conjecture that our results can be generalized to the billiards on non-compact manifolds with non-trivial metrics (also in the presence of a potential), e.g., billiards on the hyperbolic plane. In such a case, one needs to adjust the notion of domain 'convexity' to the corresponding classical dynamics. In other words, a domain should be defined as 'convex' if the interior–exterior duality holds and defined as 'non-convex' if it breaks down.

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Appendix

Proposition 3. *Let* Φ_Γ^σ, $\Phi_\Gamma^{\bar\sigma}$ *be the states:*

$$\begin{aligned} \left|\Phi_\Gamma^\sigma\right\rangle &= \frac{1}{C_1}\int_0^T \mathrm{e}^{\mathrm{i}(E-\mathsf{H}_0)t/\hbar}\left|\phi_{\mathsf{z}}^\sigma\right\rangle \mathrm{d}t \qquad \sigma = (\sigma^0, \sigma^1) \\ \left|\Phi_\Gamma^{\bar\sigma}\right\rangle &= \frac{1}{C_2}\int_0^T \mathrm{e}^{\mathrm{i}(E-\mathsf{H}_0)t/\hbar}\left|\phi_{\mathsf{z}}^{\bar\sigma}\right\rangle \mathrm{d}t \qquad \bar\sigma = (\bar\sigma^0, \bar\sigma^1) \end{aligned} \tag{A.1}$$

localized along the path $\Gamma = \Gamma(E), \Gamma(E) = \{\Psi^t \cdot \mathsf{z} = (q(t), p(t)), t \in [0, T], E = p^2/2\}$ *with* $\sigma^0 = \mathrm{i}\beta_1, \bar\sigma^0 = \mathrm{i}\beta_2$*;* $\beta_1, \beta_2 > 0$ *and* $\sigma^1 = \bar\sigma^1$*. Then*

$$\left\langle\Phi_\Gamma^\sigma\middle|\Phi_\Gamma^\sigma\right\rangle = \frac{T}{C_1^2}\left(\frac{2\pi\hbar}{\beta_1 E}\right)^{1/2} + O(\hbar) \qquad \left\langle\Phi_\Gamma^{\bar\sigma}\middle|\Phi_\Gamma^{\bar\sigma}\right\rangle = \frac{T}{C_2^2}\left(\frac{2\pi\hbar}{\beta_2 E}\right)^{1/2} + O(\hbar) \tag{A.2}$$

$$\left\langle\Phi_\Gamma^\sigma\middle|\Phi_\Gamma^{\bar\sigma}\right\rangle = \left\langle\Phi_\Gamma^\sigma\middle|\Phi_\Gamma^\sigma\right\rangle^{1/2}\left\langle\Phi_\Gamma^{\bar\sigma}\middle|\Phi_\Gamma^{\bar\sigma}\right\rangle^{1/2} + O(\hbar). \tag{A.3}$$

Proof. The inner product

$$\left\langle\Phi_\Gamma^\sigma\middle|\Phi_\Gamma^{\bar\sigma}\right\rangle = \frac{1}{C_1C_2}\int_0^T\int_0^T \left\langle\phi_{\mathsf{z}}^\sigma\right| \mathrm{e}^{\mathrm{i}(E-\mathsf{H}_0)(t_1-t_2)/\hbar}\left|\phi_{\mathsf{z}}^{\bar\sigma}\right\rangle \mathrm{d}t_1\,\mathrm{d}t_2 \tag{A.4}$$

can be written as

$$\begin{aligned} \left\langle\Phi_\Gamma^\sigma\middle|\Phi_\Gamma^{\bar\sigma}\right\rangle &= \frac{1}{2C_1C_2}\left(\int_0^T (T-t)K(t)\,\mathrm{d}t + \int_0^T (T-t)K(-t)\,\mathrm{d}t\right) \\ &= \frac{1}{2C_1C_2}\int_{-T}^T (T-|t|)K(t)\,\mathrm{d}t \end{aligned} \tag{A.5}$$

where

$$K(t) = \langle \phi_{\mathsf{z}}^{\sigma} | \, \mathrm{e}^{\mathrm{i}(E-\mathsf{H}_0)t/\hbar} \, | \phi_{\mathsf{z}}^{\bar{\sigma}} \rangle . \tag{A.6}$$

By the propagation formula (19) we get for (A.6)

$$\begin{aligned} K(t) &= \mathrm{e}^{\mathrm{i}(S(t)+E)/\hbar + \mathrm{i}\mu(t)} \, \langle \phi_{\mathsf{z}}^{\sigma} | \, \phi_{\mathsf{z}(t)}^{\bar{\sigma}(t)} \rangle + O(\hbar) \\ &= \det \left(\frac{4 \operatorname{Im} \sigma \operatorname{Im} \bar{\sigma}^*(t)}{(\sigma - \bar{\sigma}^*(t))^2} \right)^{1/4} \exp \left(-\frac{\mathrm{i}t^2}{2\hbar} \left\langle p, \bar{\sigma}^*(t) \frac{1}{\sigma - \bar{\sigma}^*(t)} \sigma p \right\rangle \right) + O(\hbar) \\ &= \left(\frac{(\beta_2 \beta_1)^{1/4}}{(\beta_2 + \beta_1)^{1/2}} + O(t) \right) \exp \left(-\frac{t^2 p^2 \beta_2 \beta_1}{2\hbar(\beta_2 + \beta_1)} + O(t^3) \right) + O(\hbar). \end{aligned} \tag{A.7}$$

After inserting this expression into equation (A.5) and applying the stationary phase approximation to the integral one gets (A.2) and (A.3). Finally, let us note that equation (A.3) remains true also when β_1 or β_2 equals zero. □

References

[1] Berry M 1977 Regular and irregular semiclassical eigenfunctions *J. Phys. A: Math. Gen.* **10** 2083
[2] Backer A and Schubert R 2002 Amplitude distribution of eigenfunctions in mixed systems *J. Phys. A: Math. Gen.* **35** 527–38
[3] Blum G, Gnutzmann S and Smilansky U 2002 Nodal domains statistics—a criterion for quantum chaos *Phys. Rev. Lett.* **88** 114101
[4] Heller E J 1991 *Chaos and Quantum Systems* ed M J Giannoni, A Voros and J Zinn-Justin (Amsterdam: Elsevier) p 548
[5] Li B and Hu B 1998 Statistical analysis of scars in stadium billiard *J. Phys. A: Math. Gen.* **31** 483–504
[6] Li B, Robnik M and Hu B 1998 *Phys. Rev.* E **57** 4095
[7] Cohen D, Lepore N and Heller E J 2001 Consolidating boundary methods for finding the eigenstates of billiards *Preprint* nlin.CD/0108014
[8] Malgrange B 1955 Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution *Ann. Inst. Fourier Grenoble* **6** 271–355
[9] Hörmander L 1983 *The Analysis of Linear Partial Differential Equations* (Berlin: Springer)
[10] Berry M 1994 Evanescent and real waves in quantum billiards, and Gaussian beams *J. Phys. A: Math. Gen.* **27** L391–8
[11] Doron E and Smilansky U 1992 Semiclassical quantization of chaotic billiards: a scattering theory approach *Nonlinearity* **5** 1055–84
[12] Eckmann J P and Pillet C A 1995 Spectral duality for planar billiards *Commun. Math. Phys.* **170** 283–313
[13] Dietz B and Smilansky U 1993 A scattering approach to the quantization of billiards—the inside–outside duality *Chaos* **3** 581–90
[14] Smilansky U 1994 Semiclassical quantization of chaotic billiards—a scattering approach *The Les Houches Summer School on Mesoscopic Quantum Physics* (Amsterdam: North-Holland) pp 373–431
[15] Babich V M 1999 The higher-dimensional WKB method or ray method. Its analogies and generalizations *Partial Differential Equations V* (Berlin: Springer)
[16] Paul T and Uribe A 1996 On the pointwise behavior of the semi-classical measures *Commun. Math. Phys.* **170** 283–313
[17] Paul T and Uribe A 1993 A construction of quasi-modes using coherent states *Ann. Inst. Poincaré* **59** 357–81
[18] Paul T 1997 *Semi-classical Methods with Emphasis on Coherent States (The IMA Volumes in Mathematics and its Applications vol 95)* (New York: Springer)
[19] Lazutkin V F 1999 Semiclassical asymptotics of eigenfunctions *Partial Differential Equations V* (Berlin: Springer)
[20] Schubert R 2001 Semiclassical localization in phase space *PhD Thesis* Universität Ulm
[21] Cardoso F and Popov G 2002 Quasimodes with exponentially small errors associated with elliptic periodic rays *Preprint* mp-arc/02-250